

# LATTICE OF RELATIONAL ALGEBRAS DEFINABLE IN INTEGERS WITH SUCCESSOR

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**ABSTRACT.** Svenonius theorem reduces the problem of first-order definability to the problem of relationship between groups of permutations. In the present paper we use this approach to describe lattices of definable relations for the structure of rational numbers with the order relation and for the structure of integer numbers with the successor relation.

Consider a structure  $M = \langle A, \Sigma \cup \{R\} \rangle$  with support  $A$  and signature  $\Sigma \cup \{R\}$ . Relation  $R$  is *definable* in  $\langle A, \Sigma \rangle$  (or simply definable by signature  $\Sigma$ ), if  $M \models (\forall x_1, \dots, x_n)(R(x_1, \dots, x_n) \equiv S(x_1, \dots, x_n))$  for some first-ordered formula  $S$  in the signature  $\Sigma$ . We denote by  $\Sigma \succcurlyeq R$  that  $R$  is definable by  $\Sigma$ , we write  $Q \succcurlyeq R$  instead of  $\{Q\} \succcurlyeq R$ . If  $Q$  and  $R$  are relations on the same set then  $Q \approx R$  denotes that  $Q \succcurlyeq R$  and  $R \succcurlyeq Q$ ,  $Q \succ R$  denotes that  $Q \succcurlyeq R$  and  $R \not\succcurlyeq Q$ . In the present paper we investigate the mutual definability of relations in some structures.

If  $\varphi$  is permutation on set  $B$  (bijectively mapping of  $B$  on  $B$ ) and  $P$  – some  $n$ -ary relation on  $B$  then we say that  $\varphi$  *preserves*  $P$ , if  $P(a_1, \dots, a_n) \equiv P(\varphi(a_1), \dots, \varphi(a_n))$  for all  $a_1, \dots, a_n \in B$ . A collection of permutations preserves a relation if all members of the collection preserve the relation.

Our main tool will be Svenonius theorem [1], which in our case may be formulated as follows:

**Svenonius theorem.** *Let  $M = \langle A, \Sigma \cup \{R\} \rangle$  be a countable structure with signature  $\Sigma \cup \{R\}$ . Relation  $R$  is definable in  $\langle A, \Sigma \rangle$  iff for any countable elementary extension  $\langle A', \Sigma \cup \{R\} \rangle$  of  $M$ , the group of permutations on  $A'$  preserving all relations from  $\Sigma$ , preserves  $R$ .*

Hereby the Svenonius theorem reduces the problem of definability of relations to the problem of relationship between groups of permutations, preserving relations. We are going to illustrate the usefulness of this tool by two simple examples.

## 1. RATIONAL NUMBERS $Q$ WITH THE ORDER RELATION $<$

The lattice of definable relations on the set of rational numbers with the order was first described in [2] by syntactical methods. Here we will get the same description with the help of Svenonius theorem.

By  $M_Q$  we denote the structure  $\langle Q, \{<\} \rangle$ . In the present section by *shift* we mean a monotone increasing permutation on  $Q$ . By  $\Gamma$  we denote the group of all shifts. Next two statements are trivial:

**Lemma 1.** *If  $a_1, \dots, a_n, b_1, \dots, b_n$  are rational numbers and  $a_i < a_j \Leftrightarrow b_i < b_j$  for  $i, j \leq n$  then partial mapping  $f(a_i) = b_i$  can be extended to a shift.*

**Lemma 2.** *If relation  $R(x_1, \dots, x_n)$  is definable in  $M_Q$  and  $a_1, \dots, a_n, b_1, \dots, b_n$  are rational numbers and  $a_i < a_j \Leftrightarrow b_i < b_j$  for  $i, j \leq n$  then  $R(\bar{a}) \equiv R(\bar{b})$ .*

We associate with any set  $\bar{a} = \{a_1, \dots, a_n\}$  of mutually different rational numbers a permutation  $\sigma$  on natural numbers  $\{1, \dots, n\}$ , where  $\sigma(i)$  is the ordinal number of  $a_i$  in ordering  $\bar{a}$ , namely  $\sigma(i) = |\{j \mid a_j \leq a_i\}|$ . It is clear, due to lemma 2, that for any relation, definable in  $M_Q$ , its value on a collection  $\bar{a}$  depends not on the set  $\bar{a}$  itself, but on the corresponding permutation  $\sigma$ . A permutation  $f$  on  $Q$  realizes permutation  $\sigma$  on  $\{1, \dots, n\}$  if permutation  $\sigma$  is equal to  $\{f(1), \dots, f(n)\}$ . A set of permutations on  $Q$  realizes  $\sigma$  if some member of the set realizes  $\sigma$ .

Any countable elementary extension of the structure  $M_Q$  is isomorphic to  $M_Q$ , so we can just consider all permutations on  $M_Q$ . If relation  $R$  on the set of rational numbers is definable in  $M_Q$  then any shift preserves  $R$ . If the order relation  $<$  is undefinable in  $\langle Q, \{R\} \rangle$  then, due to Svenonius theorem, there is a permutation  $f$  on  $Q$ , preserving  $R$ , which is not a shift. In other words, if relation  $R$  is definable in  $M_Q$  but the order relation is undefinable in  $\langle Q, \{R\} \rangle$  then the relation  $R$  is preserved by the group of permutations  $\Gamma_f$ , generated by  $\Gamma \cup \{f\}$  for some  $f \notin \Gamma$ .

A permutation  $f$  on  $Q$  (and the corresponding group  $\Gamma_f$ ) will be called *k-nontrivial*, if not every permutation on  $\{1, \dots, k\}$  is realized by  $\Gamma_f$ . A permutation  $f$  on  $Q$  (group  $\Gamma_f$ ) will be called *nontrivial*, if it is *k-nontrivial* for some natural  $k$ .

It is clear, that if  $f$  is trivial then  $\Gamma_f$  preserves only the identically true (false) relation.

**Lemma 3.** *Suppose that  $f$  is permutation on  $Q$ , collections  $\{a_1 < \dots < a_k\}$  and  $\{b_1 < \dots < b_k\}$  of rational numbers are such, that*

(i)  *$f$  monotonically increases on  $\{a_i\}$  and monotonically decreases on  $\{b_i\}$ .*

(ii)  $[a_1, a_k] \cap [b_1, b_k] = [f(a_1), f(a_k)] \cap [f(b_k), f(b_1)] = \emptyset$

*Then permutation  $f$  is  $k$ -trivial.*

*Proof.* We note that, because  $f$  monotonically decreases on  $\{b_i\}$  then  $\Gamma_f$  realizes the permutation  $\{k, k-1, \dots, 1\}$ . So if  $f$  is  $k$ -nontrivial then permutations  $-f(x), f(-x), -f(-x)$  are  $k$ -nontrivial as well. Hence we may assume without loss in generality that  $a_k < b_1$  and  $f(a_k) < f(b_k)$ .

Consider the restriction of  $f$  to  $\{a_i\} \cup \{b_i\}$ . It is clear, that for any  $i < k$  the group  $\Gamma_f$  realizes the permutation  $A_i^k = \{1, \dots, i, k, k-1, \dots, i+1\}$ . Since  $A_i^k \circ A_{i-1}^k = \{1, \dots, i-1, i+1, \dots, k, i\}$ , it is obvious, that  $f$  is  $k$ -trivial.  $\square$

**Lemma 4.** *Any nontrivial permutation on  $Q$  is continuous on  $Q$ .*

*Proof.* By contradiction. Let  $\{a_i\}$  be a sequence converging to  $a$  such as  $|f(a) - f(a_i)| > \varepsilon$  for some  $\varepsilon > 0$ . We may suppose that  $\{a_i\}$  monotonically increases, otherwise we consider the permutation  $f(-x)$ . We can suppose, by Ramsey theorem, that  $f$  is monotonic on  $\{a_i\}$ , moreover, that  $f$  monotonically increases on  $\{a_i\}$ : otherwise we consider the permutation  $-f$ .

Consider any natural  $k$  and prove that the permutation  $f$  is  $k$ -trivial.

Once more due to Ramsey theorem we can find such collection  $b_1 < b_2 < \dots < b_k$  of rational numbers that (i)  $f$  is monotonic on  $\{b_i\}$  (ii)  $0 < f(a) - f(b_i) < \varepsilon$  and (iii) or  $a > b_i, i \leq k$  or  $a < b_i, i \leq k$ . If  $f$  monotonically decreases on  $\{b_i\}$  then, by the previous lemma,  $f$  is  $k$ -trivial, so we suppose, that  $f$  monotonically increases on  $\{b_i\}$ . There are two cases:

(1)  $a < b_i, i \leq k$ . Consider restriction of  $f$  to the set  $\{a_i\} \cup \{a\} \cup \{b_i\}$ . It is clear that for any  $j < k$  the group  $\Gamma_f$  realizes permutation  $A_j^k = \{1, 2, \dots, j-1, j+1, \dots, k, j\}$ , so  $f$  is  $k$ -trivial.

(2)  $a > b_i, i \leq k$ . We may suppose that  $a_1 > b_k$ . Consider the restriction of  $f$  to the set  $\{b_i\} \cup \{a_i\}$  then it is clear that for any  $j \leq k$  the group  $\Gamma_f$  realizes permutation  $B_j^k = \{j+1, \dots, k, 1, 2, \dots, j\}$ . Consider the restriction of  $f$  to the set  $\{b_i\} \cup \{a_i\} \cup \{a\}$  then it is clear that for any  $j \leq k$  the group  $\Gamma_f$  realizes permutation  $C_j^k = \{j+1, j+2, \dots, k-1, 1, 2, \dots, j, k\}$ . But  $A_j^k = C_{k-j}^k \circ B_j^k$ , so  $f$  is  $k$ -trivial.  $\square$

**Statement 1.** *For any nontrivial permutation  $f$  on  $Q$*

- (i)  $f$  is monotonic, or
- (ii) there is irrational number  $\alpha$  such that  $f$  is monotonically increasing on the set  $\{q \mid q < \alpha\}$  and on the set  $\{q \mid q > \alpha\}$  and  $a < \alpha < b \Rightarrow f(a) > f(b)$ , or
- (iii) the permutation  $-f$  satisfies the condition (ii)

*Proof.* We may suppose that there is a convergent sequence of rational numbers on which permutation  $f$  is monotonically increasing, otherwise we consider permutation  $-f$ . Note, due to lemma 3, that in any neighborhood of any point there is an infinite subset on which  $f$  is monotonically increasing. Suppose that  $f$  is not monotonic on  $Q$ , i.e.  $f(b) < f(a)$  for some  $a < b$ . We denote  $A = \{q \mid q < b, f(q) > f(b)\}$ ,  $B = \{q \mid q > a, f(q) < f(a)\}$ . It is clear that  $a \in A, b \in B$ .

(1) We claim that if  $c < d, d \in A(c' > d', d' \in B)$  then  $c \in A(c' \in B)$ . We prove that  $f(c) > f(b)$ , the case  $c' \in B$  is absolutely similar.

Suppose that  $f(c) < f(b) < f(d)$ . Choose two such infinite subsets  $\{c_i\}$  and  $\{b_i\}$  that (i)  $f$  monotonically increases on subsets  $\{c_i\}$  and  $\{b_i\}$ , (ii)  $\{c_i\}$  and  $\{b_i\}$  are subsets of disjoint neighborhoods of points  $c$  and  $b$  respectively, and (iii)  $f(\{c_i\})$  and  $f(\{b_i\})$  are subsets of disjoint neighborhoods of points  $f(c)$ ,  $f(b)$ . Consider the restriction of  $f$  to the set  $\{c_i\} \cup \{d\} \cup \{b_i\}$ . Note that for any  $k, j, j < k$  the group  $\Gamma_f$  realizes permutation  $G_j^k = \{1, 2, \dots, j-1, j+1, \dots, k, j\}$  which contradicts the non-triviality of  $f$ .

(2) We claim that the permutation  $f$  monotonically increases on  $A$  and  $B$ . We prove the claim for the set  $A$ , the proof for  $B$  is absolutely similar.

Suppose that there are  $c, d \in A$   $c < d < b$  and  $f(c) > f(d) > f(b)$ . Choose sets  $\{c_i\}$  and  $\{b_i\}$  as in (1). Consider the restriction of  $f$  to the set  $\{c_i\} \cup \{d\} \cup \{b_i\}$ . Note that for any  $k, j, j < k$  the group  $\Gamma_f$  realizes the permutation  $H_j^k = \{j+1, \dots, k, j, 1, 2, \dots, j-1\}$  as well as the permutation  $H_j'^k = \{j+1, \dots, k, 1, 2, \dots, j-1, j\}$ . But  $G_j^k$ , defined in (1), is equal to  $H_{k+1-j}^k \circ H_j^k$ , which contradicts the non-triviality of  $f$ .

From (1) and (2) it follows that  $\{A, B\}$  defines a cut of rational numbers, this completes the proof.  $\square$

**Summary:** Besides shifts, there are three types of nontrivial permutations on  $Q$ : monotonically decreasing on  $Q$ ; monotonically increasing on each member of some cut of rational numbers; and the composition of the two. According to [2] we denote these types by **B**, **C**, **S** correspondingly. All permutations of the same type are equivalent up to shifts, so for every type there is only one corresponding group. These groups we denote by  $\Gamma_B, \Gamma_C, \Gamma_S$  in spite of some ambiguity. The group  $\Gamma_B$  is 3-nontrivial: it doesn't realize the permutation  $\{3, 1, 2\}$  (but realizes the permutation  $\{3, 2, 1\}$ ); on the contrary the group  $\Gamma_C$  doesn't realize the permutation  $\{3, 2, 1\}$  (but realizes the permutation  $\{3, 1, 2\}$ ); the group  $\Gamma_S$  is 4-nontrivial:

it doesn't realize the permutation  $\{1, 2, 4, 3\}$ . The intersection of  $\Gamma_B$  and  $\Gamma_C$  is the group of shifts. If a permutation is realizable by the group  $\Gamma_B$  then it is realizable by  $\Gamma_S$  as well, because a permutation of type **B** can be obtained from a permutation of type **S** by restriction to a member of the cut. If a permutation is realizable by the group  $\Gamma_C$  it is realizable by  $\Gamma_S$  as well, because a permutation of type **C** is a composition of permutations of types **S** and **B**.

**In terms of logic:** Any relation definable in  $M_Q$  is equivalent by definability to order relation ( $<$ ), or is preserved by one of the groups:  $\Gamma_B, \Gamma_C, \Gamma_S$ . With each of these groups we associate for some  $k$  a  $k$ -ary relation definable in  $M_Q$ . For groups  $\Gamma_B$  and  $\Gamma_C$  choose  $k = 3$ , for  $\Gamma_S$  choose  $k = 4$ . Denote corresponding relations by  $b, c, s$ . Any relation definable in  $M_Q$  is equivalent by definability to order relation, or to one of the relations  $b, c, s$  or to identically true one. Neither  $b \succcurlyeq <$  nor  $c \succcurlyeq <$ , but  $\{b, c\} \succcurlyeq <$ . Any relation  $R$  which satisfies  $b \succcurlyeq R$  and  $c \succcurlyeq R$  satisfies  $s \approx R$  or  $R \approx$  the identically true one.

## 2. INTEGER NUMBERS $Z$ WITH THE SUCCESSOR RELATION $'$

In the present section we consider the structure  $M_Z = \langle Z, \{'\} \rangle$  – integer numbers with the successor relation. We will show that the lattice of definable in  $\langle Z, \{'\} \rangle$  relations is rather simple. For any natural number  $n$  we denote relations  $x_1 - x_2 = n, x_1 - x_2 = x_3 - x_4 = n \vee x_1 - x_2 = x_3 - x_4 = -n$ , and  $|x_1 - x_2| = n$  by  $A_n, B_n, C_n$  respectively. We will demonstrate that if relation  $R$  is definable in  $\langle Z, \{'\} \rangle$  and is not identically true (false) then  $R \approx A_n$  or  $R \approx B_n$  or  $R \approx C_n$  for some natural  $n$  and  $A_n \succ B_n \succ C_n$  for any  $n$  and if  $n \neq m$  then  $A_n \succ A_m, B_n \succ B_m, C_n \succ C_m$  iff  $n$  is a divisor of  $m$ .

Unlike the previous section there exist different countable elementary extensions of the original structure, but any of them is elementary embeddable in the structure  $M_Z$  defined as follows: the support of  $M_Z$  is  $Z \times S$  where  $S$  is some countable set and the relation  $'$  is specified as  $(x, y) = (x_1, y_1)' \iff x = x_1 + 1, y = y_1$ .

So, according to Svenonius theorem we can limit our consideration to permutations on  $M_Z$ . For any  $a \in M_Z$  we denote by  $a^1(a^2)$  the first (second) component of  $a$ . Two members  $a, b \in M_Z$  are called to be in the same *galaxy*, if  $a^2 = b^2$ . For any  $a \in M_Z, z \in Z$  by  $a \pm z$  we denote the item  $(a^1 \pm z, a^2)$ . We will also need the ordered set  $Z_\infty = Z \cup \{\infty\}$ , the order on  $Z$  is natural and  $z < \infty$  for any  $z \in Z$ . We define the function of absolute value ( $||$ ) on  $Z_\infty$ : it is natural on  $Z$  and  $|\infty| = \infty$ . The subtraction function  $(-)$  maps  $M_Z \times M_Z$  on  $Z_\infty$  as follows:  $a - b = a^1 - b^1$  if  $a^2 = b^2$  and equal to  $\infty$  if  $a^2 \neq b^2$ . The expression  $a > b$  for  $a, b \in M_Z$  is simply an abbreviation for  $a - b > 0$ . If  $m$  is natural number then we call two vectors  $\bar{a}, \bar{b} \in M_Z$  of length  $n$  *m-indistinguishable*, if  $|a_i - a_j| < m$  or  $|b_i - b_j| < m$  implies  $a_i - a_j = b_i - b_j$  for any  $i, j \leq n$ .

In the present section a *permutation* is a permutation on the support of  $M_Z$ . A permutation  $f$  is called *shift* if  $f(a) - f(b) = a - b$  for any  $a, b \in M_Z$ . It is clear that the set of all permutations preserving  $'$  is the set of shifts. By  $\Gamma$  we denote the group of all shifts.

Next two lemmas are similar to lemmas 1 and 2.

**Lemma 5.** *Suppose that  $a_1, \dots, a_n, b_1, \dots, b_n \in M_Z$  are such, that  $i, j \leq n \Rightarrow a_i - a_j = b_i - b_j$ . Then the partial mapping  $f(a_i) = f(b_i)$  can be extended to a shift*

**Lemma 6.** *For any  $n$ -ary relation  $R$  definable in  $\langle Z, \{'\} \rangle$  there exists such a natural number  $w$  (called the width of relation  $R$ ) that  $R(\bar{a}) = R(\bar{b})$  holds for any two  $w$ -indistinguishable vectors  $\bar{a}, \bar{b} \in M_Z$  with length  $n$ .*

Lemma 5 is trivial. Lemma 6 is simple too, but we are giving the proof of it.

*Proof.* Consider the structure  $\langle Z, \{+, <\} \rangle$ . The relation  $R$  is definable in it, and the statement of lemma 6 can be expressed by a first-ordered formula  $P$  in  $\langle Z, \{+, <\} \rangle$ . Consider a countable non-standard extension  $M_0$  of the structure  $\langle Z, \{+, <\} \rangle$ . Note, that  $P$  is true in  $M_0$  – it is enough to pick as  $w$  a non-standard number and apply the lemma 5. So the formula  $P$  is true in  $M_0$  for some standard number as well. Since the structures  $M_0$  and  $M_Z$  are isomorphic (as structures with the only relation  $'$ ), the statement of lemma 6 holds in  $M_Z$ .  $\square$

We note, as in the previous section, that relation  $R$  is definable in  $\langle Z, \{'\} \rangle$  and the relation  $'$  is not definable by  $R$  iff  $R$  is preserved by a group of permutations  $\Gamma_f$  for some  $f$  that is not a shift, i.e.  $f(a) - f(b) \neq a - b$  for some  $a, b \in M_Z$ .

Let a group of permutations  $\Gamma'$  includes  $\Gamma$ . Two members  $z_1, z_2 \in Z_\infty$  will be called *equivalent* (respectively  $\Gamma'$ ) if  $\gamma(a) - \gamma(b) = z_2$  for some  $\gamma \in \Gamma', a, b \in M_Z, a - b = z_1$ . The equivalence class (respectively  $\Gamma'$ ) of  $z$  we denote by  $K_z$ . A number  $z \in Z$  is called *regular* (respectively  $\Gamma'$ ) if  $K_z$  is finite and  $\infty \notin K_z$ .

**Lemma 7.**

- (i) If  $z_1, z_2$  are regular numbers then  $z_1 \pm z_2$  is regular.
- (ii) Greatest common divisor of regular numbers is regular.

*Proof.* Item (i) holds by definition, item (ii) follows from (i).  $\square$

**Lemma 8.** Let a group of permutations  $\Gamma'$  includes  $\Gamma$  and  $d$  is the greatest common divisor of all regular respectively  $\Gamma'$  numbers. Then  $K_d = \{d\}$  or  $K_d = \{d, -d\}$ .

Moreover, if  $K_d = \{d\}$  then  $K_z = \{z\}$  for any  $z$  which is a multiple of  $d$ , if  $K_d = \{d, -d\}$  then  $K_z = \{z, -z\}$  for any  $z$  which is a multiple of  $d$ .

*Proof.* Denote by  $D$  a number equivalent to  $d$  with the maximum absolute value. Suppose that  $D = N \cdot d$  or  $D = -N \cdot d$  for some natural number  $N > 1$ . Choose  $\gamma \in \Gamma', a, b \in M_Z$  such as  $a - b = D, \gamma(a) - \gamma(b) = d$ . For any  $0 \leq k < N$  denote  $C_k = \{a + k \cdot d + z \cdot D \mid z \in Z\}$ . Since  $d$  is regular, the expression  $\gamma(a) - \gamma(c)$  is finite and it is a multiple of  $d$  for any  $c \in C_k, 0 \leq k < N$ . Consider the set  $E = \{\gamma(a), \gamma(a) + d, \dots, \gamma(a) + (N - 1) \cdot d\}$ . Since images of  $a, b \in C_0$  under permutation  $\gamma$  belong to  $E$  then there is  $0 \leq k' < N$  such as the intersection of  $E$  with the set  $\gamma(C_{k'})$  is empty. Since the absolute value of  $D$  is maximal in the equivalence class of  $d$  then or  $\gamma(c) < \gamma(a)$  for any  $c \in C_{k'}$ , or  $\gamma(a) + (N - 1) \cdot d < \gamma(c)$  for any  $c \in C_{k'}$ . Suppose that  $\gamma(c) < \gamma(a)$  for any  $c \in C_{k'}$  (another case is absolutely similar). There is  $0 \leq k'' < N$  such as the set  $\{c \in C_{k''} \mid \gamma(c) > \gamma(a)\}$  is infinite. Then the expression  $|\gamma(a + k' \cdot d + z \cdot D) - \gamma(a + k'' \cdot d + z \cdot D)|$  can be arbitrary big when  $z \in Z$ , which contradicts the regularity of  $(k' - k'') \cdot d$ . So  $N = 1$  and  $K_d = \{d\}$  or  $K_d = \{d, -d\}$ .

If  $K_d = \{d\}$  then it is clear that  $K_z = \{z\}$  for any  $z$  multiple of  $d$ . Suppose that  $K_d = \{d, -d\}$  and  $z = n \cdot d$  where  $n$  is a natural number (case of  $z = -n \cdot d$  is absolutely similar). For any  $0 \leq i < n$  and any  $\gamma \in \Gamma', a \in M_Z$  it holds  $\gamma(a + (i + 1) \cdot d) - \gamma(a + i \cdot d) = d$  or  $\gamma(a + (i + 1) \cdot d) - \gamma(a + i \cdot d) = -d$ . Moreover, since  $\gamma(a + (i + 2) \cdot d) \neq \gamma(a + i \cdot d)$  all the differences have the same sign, i.e.  $\gamma(a + n \cdot d) - \gamma(a) = n \cdot d$  or  $\gamma(a + n \cdot d) - \gamma(a) = -n \cdot d$  for any  $a \in M_Z$ , i.e.  $K_z \subset \{z, -z\}$ . Since there are  $\gamma \in \Gamma', a \in M_Z$ , such as  $\gamma(a + d) - \gamma(a) = -d$ , then  $\gamma(a + z) - \gamma(a) = -z$  and  $K_z = \{z, -z\}$ .  $\square$

Hereby if a group of permutations  $\Gamma'$  includes  $\Gamma$  and  $d$  is the greatest common divisor of all regular respectively  $\Gamma'$  numbers and  $f \in \Gamma'$  then there are three essential possibilities (the case when there is no regular number is trivial): (1)  $f(a + n \cdot d) - f(a) = n \cdot d$  for any  $a \in M_Z$  and any natural number  $n$  (such permutations are called permutations of *first type*), (2)  $f(a + n \cdot d) - f(a) = -n \cdot d$  for any  $a \in M_Z$  and any natural number  $n$  (such permutations are called permutations of *second type*), and (3) for any  $a \in M_Z$  and any natural number  $n$  it holds  $f(a + n \cdot d) - f(a) = n \cdot d$  or  $f(a + n \cdot d) - f(a) = -n \cdot d$ , each of this equalities is realized by some  $a, n$  (such permutations are called permutations of *third type*).

If  $K_d = \{d\}$  then any permutation  $f \in \Gamma'$  belongs to first type. If  $K_d = \{d, -d\}$  then a permutation  $f \in \Gamma'$  may belongs to first, second or third type.

From now by  $\Gamma_R$  we denote the set of permutations preserving relation  $R$ .

The point of the following lemma is that if some differences between items of a vector  $\bar{a}$  are non-regular (respectively  $\Gamma_R$ ) then they can be changed to infinity without changing the value of  $R(\bar{a})$ .

**Lemma 9.** *For any definable in  $\langle Z, \{\cdot\} \rangle$  relation  $R(x_1, \dots, x_n)$  and a vector  $\bar{a} = (a_1, \dots, a_n) \in M_Z$  we can find a vector  $\bar{b} = (b_1, \dots, b_n) \in M_Z$  such that*

- (i)  $R(\bar{a}) = R(\bar{b})$ ;
- (ii) if the difference  $a_i - a_j$  is not regular (respectively  $\Gamma_R$ ) then  $b_i - b_j = \infty$ .
- (iii) if the difference  $a_i - a_j$  is regular then  $|a_i - a_j| = |b_i - b_j|$ . Moreover, if  $\Gamma_R$  contains permutations of the first type only then  $a_i - a_j = b_i - b_j$ ; if  $\Gamma_R$  doesn't contain permutations of the third type then or  $a_i - a_j = b_i - b_j$  for any  $i, j$  with regular difference  $a_i - a_j$  or  $a_i - a_j = b_j - b_i$  for any  $i, j$  with regular difference  $a_i - a_j$ ;

*Proof.* We prove by induction on number of pairs  $i, j$  such that  $a_i - a_j \neq \infty$  and  $a_i - a_j$  is not a regular number. Suppose that  $a_i - a_j \neq \infty$  and it is not regular. Let  $w$  be the width of relation  $R$  and  $w'$  is the maximal absolute value of regular differences  $|a_k - a_l|$ ,  $(k, l \leq n)$ . Then we can find such  $\gamma \in \Gamma_R$  that  $|\gamma(a_i) - \gamma(a_j)| > n \cdot \max(w, w')$ .

First of all we find such a vector  $\bar{a}'$  that (1)  $a'_i - a'_j = \gamma(a_i) - \gamma(a_j)$ , (2) if the difference  $a_k - a_l$ ,  $(k, l \leq n)$  is regular then  $a'_k - a'_l$  is regular, and (3) if  $a_k$  and  $a_l$ ,  $(k, l \leq n)$  belong to different galaxies then  $a'_k$  and  $a'_l$  belong to different galaxies as well. To do that we apply to  $\bar{a}$  such a shift  $s$  that  $s(a_i) = a_i$ ,  $s(a_j) = a_j$  and if  $a'_k$  and  $a'_l$  belong to different galaxies then  $|\gamma(s(a_k)) - \gamma(s(a_l))| > w$ . After that we can use lemma 6 to find such vector  $\bar{a}'$  that  $R(\bar{a}') = R(\gamma(s(\bar{a}))) = R(\bar{a})$ .

If  $a'_i - a'_j = \infty$  then  $\bar{b} = \bar{a}'$ : the number of pairs  $i, j$  with non-regular difference  $b_i - b_j \neq \infty$  decreased.

If  $a'_i - a'_j < \infty$  then  $a'_i, a'_j$  belong to the same galaxy  $U$ . Suppose that  $a'_i < a'_j$ . Let  $c_1 < \dots < c_k$  be all items of vector  $\bar{a}'$  from galaxy  $U$ . There is such  $c_l$ , that  $a'_i \leq c_l < c_{l+1} \leq a'_j$  and  $c_{l+1} - c_l > \max(w, w')$ . We can apply to  $\bar{a}'$  such a permutation  $f$  that (1)  $f(x) = x$  if  $x \notin U$  or  $x = c_m$ ,  $m \leq l$  (2) items  $f(c_i)$ ,  $m > l$  belong to a galaxy which does not contain items from  $\bar{a}'$  and (3)  $c_m - c_{m'} = f(c_m) - f(c_{m'})$ ,  $l < m, m'$ . Denote vector  $f(\bar{a}')$  by  $\bar{b}$ . By lemma 6  $R(\bar{a}') = R(\bar{b})$  holds. Since  $c_{l+1} - c_l > w'$  then  $b_m - b_{m'}$  is regular iff  $c_m - c_{m'}$  is regular, i.e. when  $a_m - a_{m'}$  is regular number ( $1 \leq m, m' \leq n$ ). The number of pairs  $i, j$  with non-regular difference  $b_i - b_j \neq \infty$  decreased.  $\square$

**Corollary.** *If a relation  $R$  is definable in  $\langle \mathbb{Z}, \{'\} \rangle$  and no number is regular respectively  $\Gamma_R$  then  $R$  is identically true (false).*

Recall that for any natural number  $n$  by  $A_n$  we denote relation  $x_1 - x_2 = n$ , by  $B_n$  we denote relation  $x_1 - x_2 = x_3 - x_4 = n \vee x_1 - x_2 = x_3 - x_4 = -n$ , and by  $C_n$  we denote relation  $|x_1 - x_2| = n$ .

**Statement 2.** *Suppose that relation  $R$  is definable in  $\langle \mathbb{Z}, \{'\} \rangle$ ,  $d$  is the common greatest divisor of regular respectively  $\Gamma_R$  numbers. Then*

- (i) *if  $\Gamma_R$  doesn't contain permutations of second or third types then  $R \approx A_d$ .*
- (ii) *if  $\Gamma_R$  doesn't contain permutations of third type but contains a permutation of second type then  $R \approx B_d$ .*
- (iii) *if  $\Gamma_R$  contains a permutation of third type then  $R \approx C_d$ .*

*Proof.* It is clear that  $R \succcurlyeq A_d$  ( $R \succcurlyeq B_d, R \succcurlyeq C_d$  respectively): it is easy to note that if (i) holds then any permutation from  $\Gamma_R$  preserves  $A_d$ , if (ii) holds then any permutation from  $\Gamma_R$  preserves  $B_d$ , and if (iii) holds then any permutation from  $\Gamma_R$  preserves  $C_d$ .

To prove the reverse sentence we need to show that any permutation preserving  $A_d(B_d, C_d)$  belongs, if the corresponding condition holds, to  $\Gamma_R$ .

Denote by  $\Gamma'$  the set of such permutations  $f$  that  $|f(a+d) - f(a)| = d$  and  $|f^{-1}(a+d) - f^{-1}(a)| = d$  for any  $a \in M_{\mathbb{Z}}$ . It is clear that  $\Gamma'$  is the set of all permutations preserving  $C_d$ ; the subset of  $\Gamma'$  containing permutations of the first and second type is the set of all permutations preserving  $B_d$ ; the subset of  $\Gamma'$  containing permutations of the first type is the set of all permutations preserving  $A_d$ .

*Proof (i).* Suppose that there is a permutation  $f$  of the first type in  $\Gamma' \setminus \Gamma_R$ . Then there is such a vector  $\bar{a} = (a_1, \dots, a_n) \in M_{\mathbb{Z}}$  that  $R(\bar{a}) \neq R(f(\bar{a}))$ . By definition of  $\Gamma'$  if  $a_i - a_j$  is non-regular respectively  $\Gamma_R$  then the difference  $f(a_i) - f(a_j)$  is non-regular as well; by definition of the first type if  $a_i - a_j$  is regular respectively  $\Gamma_R$  then  $f(a_i) - f(a_j) = a_i - a_j$ . We use lemma 9 to find vectors  $\bar{b}, \bar{c}$  corresponding to vectors  $\bar{a}$  and  $f(\bar{a})$ . Since  $\Gamma_R$  contains permutations of the first type only, regular differences in vectors  $\bar{b}, \bar{c}$  are the same. So, according lemma 6  $R(\bar{b}) = R(\bar{c})$  which contradict to  $R(\bar{a}) \neq R(f(\bar{a}))$ .

*Proof (ii).* Suppose that there is a permutation  $f$  of the first or second type in  $\Gamma' \setminus \Gamma_R$ . We choose vectors  $\bar{a}, f(\bar{a}), \bar{b}, \bar{c}$  as in the case (i). Permutation  $f$  is the permutation of the first or second type, so if  $b_i - b_j = c_i - c_j$  for some regular difference  $b_i - b_j$  then the same equality holds for any regular difference. This contradicts the lemma 6. If  $b_i - b_j = c_j - c_i$  for some regular difference  $b_i - b_j$  then the same equality holds for any regular difference. There is a permutation  $\gamma$  of second type in  $\Gamma_R$ , i.e.  $\gamma(t+z \cdot d) - \gamma(t) = -z \cdot d$  for any  $z \in \mathbb{Z}, t \in M_{\mathbb{Z}}$ . Choose a shift  $\varphi$ , such that if  $c_i$  and  $c_j$  belong to different galaxies then  $\gamma(\varphi(c_i))$  and  $\gamma(\varphi(c_j))$  belong to different galaxies as well. Denote  $\bar{c}' = \gamma(\varphi(\bar{c}))$ , then  $b_i - b_j = c'_i - c'_j$  for any regular difference  $b_i - b_j$ . This contradicts lemma 6.

*Proof (iii).* Suppose that there is a permutation  $f$  in  $\Gamma' \setminus \Gamma_R$ . We choose vectors  $\bar{a}, f(\bar{a}), \bar{b}, \bar{c}$  as in the case (i). If the difference  $b_i - b_j$  is regular then  $b_i - b_j = c_i - c_j$  or  $b_i - b_j = c_j - c_i$ . All differences between items of  $\bar{b}$  from the same galaxy are regular, so if  $b_i - b_j = c_i - c_j$  and  $b_k - b_l = c_k - c_l$  then  $b_i$  and  $b_k$  belong to different galaxies. There is a permutation  $\gamma$  of third type in  $\Gamma_R$ , i.e.  $\gamma(t_1 + z \cdot d) - \gamma(t_1) = z \cdot d, \gamma(t_2 + z \cdot d) - \gamma(t_2) = -z \cdot d$  holds for some  $t_1, t_2 \in M_{\mathbb{Z}}$  and any  $z \in \mathbb{Z}$ . Choose a permutation  $\varphi$ , such that (1) vector  $\varphi(\bar{c}) \in \{t_1 + z \cdot d \mid z \in \mathbb{Z}\} \cup \{t_2 + z \cdot d \mid z \in \mathbb{Z}\}$  (2) if  $c_i$  and  $c_j$  belong to the same galaxy then  $c_i - c_j = \varphi(c_i) - \varphi(c_j)$  (3) if  $c_i$  and  $c_j$  belong to different galaxies then  $|\varphi(c_i) - \varphi(c_j)| > w$

as well as  $|\gamma(\varphi(c_i)) - \gamma(\varphi(c_j))| > w$  where  $w$  is the width of the relation  $R$  (4) if a galaxy contains such  $c_i, c_j$  that  $b_i - b_j = c_i - c_j$  then  $\varphi$  maps the galaxy into  $\{t_1 + z \cdot d \mid z \in Z\}$  (5) if a galaxy contains such  $c_i, c_j$  that  $b_i - b_j = c_j - c_i$  then  $\varphi$  maps the galaxy into  $\{t_2 + z \cdot d \mid z \in Z\}$ . According to lemma 6  $R(\varphi(\bar{c})) = R(\bar{c})$  holds. Denote  $\bar{c}' = \gamma(\varphi(\bar{c}))$ , then  $b_i - b_j = c'_i - c'_j$  for any regular difference  $b_i - b_j$ . This contradicts the lemma 6.  $\square$

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